Optimization of Shallow Trusses Against Limit Point Instability

M. P. Kamat*
Virginia Polytechnic Institute and State University, Blacksburg, Virginia
and
N. S. Khot† and V. B. Venkayya†
Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio

The problem of the maximization of the critical load (limit point instability) of shallow space trusses of constant volume or weight is considered. Exact solutions being possible in simple cases, a detailed investigation of the question of the uniqueness of the optimum designs for two- and four-bar shallow truss problems is carried out. The optimized trusses satisfy the constant-strain energy density criteria and the load maximization problem is verified to be the dual of the minimization of the total potential energy.

Introduction

PREVIOUS investigations with regard to optimization of shallow trusses appear to have been exclusively with regard to changes in geometry.[1] The emphasis in Ref. 1 was on demonstrating the equivalence of stability and optimality conditions in the optimal shape of shallow trusses of minimum weight. Earlier, Stadler[2] provided some observations concerning the stability of optimum shapes of uniform shallow arches of minimum weight and minimum maximum deflection obtained on the basis of a purely linear analysis.

A problem identical in objective to the one being addressed here has been investigated by Christensen[3] and Caldwell[4] for the case of a shallow elastic arch. Because of the complexity of the problem, both Christensen and Caldwell were able to provide only approximate numerical solutions of the problem.

One of the two problems considered in the present paper, namely the two-bar truss, is simple enough to yield an exact solution. The formulation of the four-bar problem provides explicit expressions for the objective function and the constraint as functions of the design variables. But, the expressions for the general configuration of the truss become complicated enough to defy an exact optimum solution without recourse to symbolic manipulation. However, utilizing the single-equality constraint, the problem can be easily converted into an equivalent unconstrained problem that can then be solved by any appropriate algorithm for unconstrained optimization. Alternatively, an algorithm of constrained optimization could equally be used to solve the original constrained problem directly.

The focus of this paper is thus on optimizing simple space trusses that permit solutions in terms of closed-form expressions or others that permit such solutions through a combination of analytical and numerical schemes. The objective is to obtain an improved understanding of the problem intricacies that will serve as a basis for optimizing complicated space trusses under more general loading and boundary conditions as well as constraints.

Received Oct. 15, 1982; revision submitted April 25, 1983.

*Professor, Department of Engineering Science and Mechanics. (This work was completed while visiting FDL on a year's IPA assignment under AFOSR's University Resident Research Program.)
†Aerospace Engineer, Structures and Dynamics Division.

The Two-Bar Truss

Evaluation of the Critical Load

The shallow, symmetrical two-bar truss of Fig. 1a is probably the most common structural model used in textbooks on elastic stability[5] for illustrating the concept of the limit point instability. The optimum configuration of this model under the constant-volume constraint is known trivially, namely

$$A_1 = A_2 = V_0/L \tag{1}$$

where V_0 is the specified volume and L the length of each of the two members of the symmetrical truss. It is this trivial nature of the solution for a symmetrical shallow truss that suggests the consideration of an unsymmetrical shallow truss to obtain a more meaningful and a nontrivial solution.

Consider the truss shown in Fig. 1b under the assumptions that

$$(H/L_1)^2 \ll 1; \qquad L_1 < L_2 \tag{2}$$

and u and v displacements positive as shown. The strains ϵ_1 and ϵ_2 in the two members can be written as[6]

$$\epsilon_1 = \frac{\sqrt{(L_1+v)^2+(H-u)^2} - \sqrt{L_1^2+H^2}}{\sqrt{L_1^2+H^2}} \tag{3a}$$

$$\epsilon_2 = \frac{\sqrt{(L_2-v)^2+(H-u)^2} - \sqrt{L_2^2+H^2}}{\sqrt{L_2^2+H^2}} \tag{3b}$$

Letting

$$L_1/L_2 = \gamma, \quad u/L_i = \zeta_i, \quad v/L_i = \eta_i, \quad \text{and} \quad H/L_i = \alpha_i$$

and using Eq. (2), Eqs. (3a) and (3b) are simplified to

$$\epsilon_1 = -\alpha_1\zeta_1 + \tfrac{1}{2}\zeta_1^2 + \eta_1 \tag{3c}$$

$$\epsilon_2 = (-\alpha_1\zeta_1 + \tfrac{1}{2}\zeta_1^2 - \eta_1/\gamma)\gamma^2 \tag{3d}$$

Extremization of the total potential energy of the system

$$\pi = \tfrac{1}{2}EA_1L_1\epsilon_1^2 + \tfrac{1}{2}EA_2L_2\epsilon_2^2 - PL_1\zeta_1$$

with respect to ζ_1 and η_1 renders the following two equations of equilibrium:

$$(m+\gamma^3)\left(\alpha_1^2\zeta_1 - \frac{3}{2}\alpha_1\zeta_1^2 + \frac{1}{2}\zeta_1^3\right) + \eta_1(m-\gamma^2)(-\alpha_1+\zeta_1) = \bar{P} \tag{4a}$$

and

$$(m-\gamma^2)(-\alpha_1\zeta_1+\tfrac{1}{2}\zeta_1^2)+\eta_1(m+\gamma)=0 \tag{4b}$$

where $\bar{P}=P/EA_2$ and $m=A_1/A_2$. Substituting the solution for η_1 from Eq. (4b)

$$\eta_1=\frac{(m-\gamma^2)(\alpha_1\zeta_1-\tfrac{1}{2}\zeta_1^2)}{(m+\gamma)} \tag{5}$$

in Eq. (4a) finally yields

$$\bar{P}=\left(\alpha_1^2\zeta_1-\frac{3}{2}\alpha_1\zeta_1^2+\frac{1}{2}\zeta_1^3\right)\frac{m\gamma(\gamma+1)^2}{(m+\gamma)} \tag{6}$$

The critical value of $\bar{P}$ is obtained by setting $\mathrm{d}\bar{P}/\mathrm{d}\zeta_1=0$. This yields

$$\alpha_1^2-3\alpha_1\zeta_1+\frac{3}{2}\zeta_1^2=0 \tag{7}$$

Of the two roots $(\zeta_1)_{1,2}=\alpha_1[1\pm(1/\sqrt{3})]$, the one that corresponds to $\bar{P}_{cr}$ is $\zeta_1=\alpha_1[1-(1/\sqrt{3})]$. With this, $\bar{P}_{cr}$ is easily verified to be

$$\bar{P}_{cr}=\frac{m\gamma(\gamma+1)^2}{(m+\gamma)}\frac{\alpha_1^3}{3\sqrt{3}}$$

or

$$P_{cr}=\frac{EA_1A_2\gamma(\gamma+1)^2}{(A_1+A_2\gamma)}\frac{\alpha_1^3}{3\sqrt{3}} \tag{8}$$

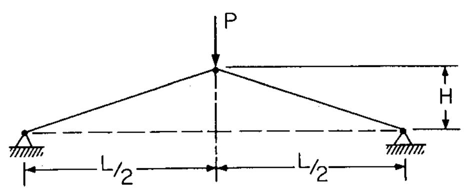

Fig. 1a Shallow symmetrical two-bar truss.

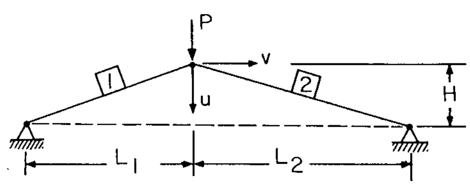

Fig. 1b Shallow unsymmetrical two-bar truss.

Table 1 Summary of results for the two-bar truss

Input: Design information
Material: aluminum
Modulus of elasticity: $E=10^7$ psi
Specified volume: $V_0=200$ in.^{3}
$L_1=200$ in., $L_2=50$ in., $H=2.50$ in. (see Fig. 1b)
Initial areas: $A_1=0.952381$ in.^{2}, $A_2=0.190476$ in.^{2}

Output: Design information
Critical load: $P_{cr}=59.981304$ lb
Potential energy: $\pi=-20.08993$ lb-in.
Displacements: $u=1.056624$ in., $v=-0.03119883$ in.
Measure of strain energy density: $\epsilon_1=\epsilon_2=-0.208057\times10^{-3}$

Method	Final areas, in.^{2} A_1	A_2
Unconstrained minimization A_1 independent design variable	0.7997502	0.7997502
VMCON	0.7997506	0.7997486

Optimization for Constant Volume

The objective is to maximize P_{cr} subject to the constraint

$$A_1L_1+A_2L_2=V_0=\text{specified const}$$

Thus, it is necessary that $\partial P_{cr}^*/\partial A_i=0$ where

$$P_{cr}^*=\frac{EA_1A_2\gamma(\gamma+1)^2}{(A_1+A_2\gamma)}\frac{\alpha_1^3}{3\sqrt{3}}-\lambda[A_1L_1+A_2L_2-V_0]$$

λ being the undetermined Lagrange multiplier. The solution of the two equations

$$\frac{\partial P_{cr}^*}{\partial A_1}=\frac{E\gamma(\gamma+1)^2\alpha_1^3}{3\sqrt{3}}\left[\frac{A_2^2\gamma}{(A_1+A_2\gamma)^2}\right]-\lambda L_1=0 \tag{9}$$

$$\frac{\partial P_{cr}^*}{\partial A_2}=\frac{E\gamma(\gamma+1)^2\alpha_1^3}{3\sqrt{3}}\left[\frac{A_1^2}{(A_1+A_2\gamma)^2}\right]-\lambda L_2=0 \tag{10}$$

by eliminating λ yields

$$A_1=A_2 \tag{11}$$

as the optimality condition. This is quite an unexpected result of an unsymmetrical two-bar truss.

Next, consider the strains ϵ_1 and ϵ_2 corresponding to any equilibrium configuration. It is easily verified from Eqs. (3) and (5) that

$$\epsilon_1=\frac{\gamma(\gamma+1)}{(m+\gamma)}(-\alpha_1\zeta_1+\tfrac{1}{2}\zeta_1^2)$$

$$\epsilon_2=\frac{m\gamma(\gamma+1)}{(m+\gamma)}(-\alpha_1\zeta_1+\tfrac{1}{2}\zeta_1^2)$$

or that

$$\epsilon_1=m\epsilon_2 \tag{12a}$$

Use of the optimality condition [Eq. (11)] implies that for the optimum configuration with $A_1=A_2$ or $m=1$,

$$\epsilon_1=\epsilon_2 \tag{12b}$$

Since, the strain energy density in a typical truss member i is

$$\frac{U_i}{V_i}=\frac{U_i}{A_iL_i}=\frac{\tfrac{1}{2}EA_iL_i\epsilon_i^2}{A_iL_i}=\frac{E\epsilon_i^2}{2} \tag{12c}$$

Equation (12b) is equivalent to a statement of constant-strain energy density in each of the two bars.

Incidentally, Eq. (11) is also the optimality condition that is obtained by minimizing the volume or weight of the truss designed to have a specified limit load P_0.

Table 1 shows the details of the optimum design for an unsymmetrical two-bar truss with $\gamma=4$ and of total specified volume of 200 in.^{3}. The solution of the following optimization problem

$$\text{Maximize}\quad P_{cr}=\frac{EA_1A_2\gamma(\gamma+1)^2}{(A_1+A_2\gamma)}\frac{\alpha_1^3}{3\sqrt{3}} \tag{13a}$$

$$\text{Subject to}\quad A_1L_1+A_2L_2=V_0=200 \tag{13b}$$

is obtained using the techniques of mathematical programming. An initial guess of $m=5$ with $A_1=0.952381$ in.^{2} and $A_2=0.190476$ in.^{2} consistent with the volume constraint is used to begin the optimization procedure. As regards the specifics of the optimization procedure, the following two options are considered:

1) Converting the constrained problem into an equivalent unconstrained problem by using the volume constraint to reduce the number of independent variables by one. The ensuing maximization problem is then solved using the BFGS (Broyden-Fletcher-Goldfarb-Shanno) algorithm[7] for unconstrained optimization of a multivariate function. Positiveness of the design variables can be guaranteed by using their squares as variables for optimization together with an appropriate step length control if required.

2) Using a standard nonlinear programming algorithm such as VMCON[8] for constrained optimization. Briefly, VMCON uses a BFGS algorithm for constrained optimization. The latter consists of the solution of a sequence of quadratic programming subproblems, the details of which may be found in Ref. 8.

The performances of the two options are summarized in Table 1. It is interesting to note that a symmetrical truss with $L_1 = L_2 = 125$ in., $H = 2.5$ in., and a total volume of 200 in.^{3} yields a critical load of only 49.2573 lb [see Eq. (8)]. Comparing this with the critical load of the optimum unsymmetrical configuration of Table 1 the superiority of the unsymmetrical configuration is obvious, although unexpected.

Indeed, the exact solution for the two-bar truss is known without recourse to any of the above options for optimization; however, the effectiveness of the two options is quite evident for the four-bar space truss problem to be discussed next, wherein a closed-form solution for the general case is not as easily available.

Four-Bar Space Truss

Evaluation of the Critical Load

Consider the four-bar space truss of Fig. 2 under the action of a vertically downward load P at joint 5 with three displacement degrees of freedom. As before the assumptions

$$(H/\ell_i)^2 \ll 1; \qquad i = 1,2,3,4 \tag{14}$$

are assumed to hold.

Proceeding as in the case of the two-bar truss and using the notation

$$\xi_i = \frac{u}{\ell_i}, \quad \eta_i = \frac{v}{\ell_i}, \quad \rho_i = \frac{w}{\ell_i}, \quad \alpha_i = \frac{H}{\ell_i}$$

$$c_{ij} = \frac{a_i}{\ell_j}, \quad s_{ij} = \frac{b_i}{\ell_j}, \quad \gamma_{ij} = \frac{\ell_i}{\ell_j}, \quad \text{and} \quad m_{ij} = \frac{A_i}{A_j}$$

it is easy to verify that

$$\begin{aligned}
\epsilon_1 &= c_{11}\xi_1 + s_{11}\eta_1 - \alpha_1\rho_1 + \tfrac{1}{2}\rho_1^2 \\
\epsilon_2 &= \gamma_{12}^2(c_{11}\xi_1 - s_{21}\eta_1 - \alpha_1\rho_1 + \tfrac{1}{2}\rho_1^2) \\
\epsilon_3 &= \gamma_{13}^2(-c_{21}\xi_1 - s_{21}\eta_1 - \alpha_1\rho_1 + \tfrac{1}{2}\rho_1^2) \\
\epsilon_4 &= \gamma_{14}^2(-c_{21}\xi_1 + s_{11}\eta_1 - \alpha_1\rho_1 + \tfrac{1}{2}\rho_1^2)
\end{aligned} \tag{15}$$

Extremization of the total potential energy

$$\pi = (E/2)(A_1L_1\epsilon_1^2 + A_2L_2\epsilon_2^2 + A_3L_3\epsilon_3^2 + A_4L_4\epsilon_4^2) - PL_1\rho_1$$

with respect to ξ_1, η_1, and ρ_1 yields the three equations of equilibrium

$$\xi_1\bar{A}_1 + \eta_1\bar{B}_1 = (\alpha_1\rho_1 - \tfrac{1}{2}\rho_1^2)\bar{A}_3 \tag{16a}$$

$$\xi_1\bar{A}_2 + \eta_1\bar{B}_2 = (\alpha_1\rho_1 - \tfrac{1}{2}\rho_1^2)\bar{B}_3 \tag{16b}$$

$$E[A_1\epsilon_1 + A_2\gamma_{12}^2\epsilon_2 + A_3\gamma_{13}^3\epsilon_3 + A_4\gamma_{14}^3\epsilon_4](-\alpha_1 + \rho_1) = P \tag{16c}$$

where

$$\begin{aligned}
\bar{A}_1 &= E\gamma_{12}(A_1c_{11}^2 + A_2\gamma_{12}^3c_{11}^2 + A_3\gamma_{13}^3c_{21}^2 + A_4\gamma_{14}^3c_{21}^2) \\
\bar{A}_2 &= E\gamma_{12}(A_1s_{11}c_{11} - A_2\gamma_{12}^3s_{21}c_{11} + A_3\gamma_{13}^3s_{21}c_{21} \\
&\quad - A_4\gamma_{13}^3s_{11}c_{21}) = \bar{B}_1 \\
\bar{A}_3 &= E\gamma_{12}(A_1c_{11} + A_2\gamma_{12}^3c_{11} - A_3\gamma_{13}^3c_{21} - A_4\gamma_{14}^3c_{21}) \\
\bar{B}_2 &= E\gamma_{12}(A_1s_{11}^2 + A_2\gamma_{12}^3s_{21}^2 + A_3\gamma_{13}^3s_{21}^2 + A_4\gamma_{14}^3s_{11}^2) \\
\bar{B}_3 &= E\gamma_{12}(A_1s_{11} - A_2\gamma_{12}^3s_{21} - A_3\gamma_{13}^3s_{21} + A_4\gamma_{14}^3s_{11})
\end{aligned} \tag{17}$$

Equations (16a) and (16b) can be solved to express ξ_1 and η_1 in terms of ρ_1 as

$$\xi_1 = \frac{(\bar{A}_3\bar{B}_2 - \bar{B}_3\bar{A}_2)}{(\bar{A}_1\bar{B}_2 - \bar{A}_2\bar{B}_1)}(\alpha_1\rho_1 - \tfrac{1}{2}\rho_1^2) = -A^*(-\alpha_1\rho_1 + \tfrac{1}{2}\rho_1^2) \tag{18a}$$

$$\eta_1 = \frac{(\bar{B}_3\bar{A}_1 - \bar{A}_3\bar{B}_1)}{(\bar{A}_1\bar{B}_2 - \bar{A}_2\bar{B}_1)}(\alpha_1\rho_1 - \tfrac{1}{2}\rho_1^2) = -B^*(-\alpha_1\rho_1 + \tfrac{1}{2}\rho_1^2) \tag{18b}$$

Substitution of Eqs. (18) into Eq. (16c) yields an expression for P as

$$P = (-\alpha_1 + \rho_1)(-\alpha_1\rho_1 + \tfrac{1}{2}\rho_1^2)\bar{c} \tag{19}$$

where $\bar{c} = [E(A_1 + A_2\gamma_{12}^3 + A_3\gamma_{13}^3 + A_4\gamma_{14}^3) - A^*\bar{A}_3 - B^*\bar{B}_3]$. As before the critical value of P is obtained by setting

$$\frac{dP}{d\rho_1} = \left(\alpha_1^2 - 3\alpha_1\rho_1 + \frac{3}{2}\rho_1^2\right)\bar{c} = 0$$

Since $\bar{c}$ is a constant involving geometry and section properties, it follows that of the two roots

$$(\rho_1)_{1,2} = \alpha_1[1 \pm (1/\sqrt{3})] \tag{20a}$$

$\rho_1 = \alpha_1[1 - (1/\sqrt{3})]$ corresponds to P_{cr}, which is given by

$$P_{cr} = (\alpha_1^3/3\sqrt{3})\bar{c} \tag{20b}$$

Note the similarity with the two-bar truss.

Optimization for Constant Volume

Maximization of P_{cr} subject to the constant-volume constraint reduces to an unconstrained maximization of the function

$$P_{cr}^* = \frac{\alpha_1^3}{3\sqrt{3}}\bar{c} - \lambda\left(\sum_{i=1}^{4} A_iL_i - V_0\right) \tag{21}$$

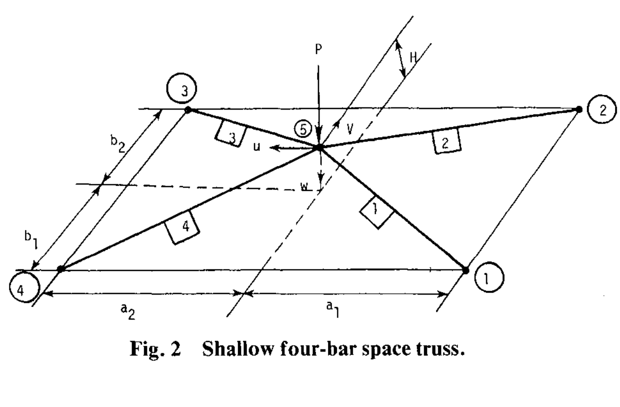

Fig. 2 Shallow four-bar space truss.

Table 2 Summary of results for the four-bar truss

Input: Design information
Material: aluminum
Modulus of elasticity: $E = 10^7$ psi
Specified volume: $V_0 = 1000$ in.^{3}
$a_1 = 60$ in., $a_2 = 144$ in., $b_1 = 120$ in., $b_2 = 72$ in., $H = 3$ in. (see Fig. 2)
Initial areas: $A_1 = A_2 = A_3 = A_4 = 1.734715347$ in.^{2}

Output: Design information (common to cases 1-4)
Critical load: $P_{cr} = 173.836966$ lb
Potential energy: $\pi = -0.69869212 \times 10^2$ lb-in.
Displacements: $u = 1.06701836 \times 10^{-2}$ in.,
$v = -6.09724775 \times 10^{-3}$ in., $w = 1.26794919$ in
Measure of strain energy density: $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = -1.735207376 \times 10^{-4}$

Case No.	Final areas, in.^{2} A_1	A_2	A_3	A_4
1	2.08120319	2.37427344	1.74393784	1.15886377
2	1.81850113	2.55783731	1.42871944	1.52585058
3	1.82744534	2.55158752	1.43945167	1.51335579
4	1.93979485	2.47308296	1.57426079	1.35640693
VMCON[a]	1.89693644	2.50303188	1.52279289	1.41631456

[a] The algorithm failed to converge to the same level of accuracy as the other four cases.

Table 3 Strain energy density distribution, four-bar truss, case 3

Iteration No.	Independent design variables, in.^{2} A_1	A_2	A_4	Measure of strain energy density, $\lvert\epsilon_i\rvert \times 10^3$ Element 1	Element 2	Element 3	Element 4
0	1.73471535	1.73471535	1.73471535	0.179193974	0.239884830	0.14317834	0.137438278
1	1.99211782	2.33298010	1.68688815	0.150177927	0.192547394	0.186933179	0.166694421
2	1.84285785	2.46719206	1.54844951	0.171384733	0.179701090	0.173054422	0.170535896
3	1.82906418	2.55152614	1.51430857	0.173295988	0.173560381	0.173704636	0.173531328
4	1.82749010	2.55158131	1.51338350	0.173514526	0.173522139	0.173525700	0.173520865
5	1.82744537	2.55158749	1.51335582	0.173520731	0.173520739	0.173520739	0.173520735
6	1.82744534	2.55158752	1.51335570	0.173520736	0.173520736	0.173520736	0.173520736
7	1.82744534	2.55158752	1.51335579	0.173520736	0.173520736	0.173520736	0.173520736
8	1.82744534	2.55158752	1.51335579	0.173520736	0.173520736	0.173520736	0.173520736

Table 4 Progress of a typical function minimization, two-bar truss

Iteration No.	Independent design variable A_1, in.^{2}	$F = -P_{cr}$, lb	$\lVert\nabla F\rVert \times 10^{-2}$, psi	π = potential energy, lb-in.
0	0.952381	−39.604041	0.86891218×10^1	−13.26484
1	0.7392856	−58.864420	0.5766399	−19.71584
2	0.7517608	−59.249888	0.4883243	−19.84495
3	0.8226244	−59.766894	0.3559970	−20.01811
4	0.7923566	−59.961374	0.9468544×10^{-1}	−20.08325
5	0.7986686	−59.980868	0.1439965×10^{-1}	−20.08978
6	0.7998033	−59.981303	0.7118583×10^{-3}	−20.08993
7	0.7997498	−59.981304	0.5053814×10^{-5}	−20.08993
8	0.7997502	−59.981304	0.1758962×10^{-8}	−20.08993
9	0.7997502	−59.981304	0.161559×10^{-13}	−20.08993

Table 5 Progress of a typical function minimization, four-bar truss, case 3

Iteration No.	Independent design variables, in.^{2} A_1	A_2	A_4	$F = -P_{cr}$, lb	$\lVert \nabla F \rVert \times 10^{-2}$, psi	π = potential energy, lb-in.
0	1.73471535	1.73471535	1.73471535	−165.72822	0.580824568	−66.6101127
1	1.99211782	2.33298010	1.68788815	−172.24834	0.339368266	−69.230705
2	1.84285785	2.46719206	1.54844951	−173.76421	0.04774763973	−69.8399696
3	1.82906418	2.55152614	1.51430857	−173.83685	0.0034866503	−69.8691641
4	1.82749010	2.55158131	1.51338350	−173.83697	$0.95142678 \times 10^{-4}$	−69.8692120
5	1.82744537	2.55158749	1.51335582	−173.83697	0.7066544×10^{-7}	−69.8692120
6	1.82744534	2.55158752	1.51335579	−173.83697	0.1893708×10^{-9}	−69.8692120
7	1.82744534	2.55158752	1.51335579	−173.83697	$0.23604514 \times 10^{-11}$	−69.8692120
8	1.82744534	2.55158752	1.51335579	−173.83697	$0.1214165 \times 10^{-13}$	−69.8692120

with respect to the areas A_i. Because of the complexity of $\bar{c}$, explicit optimality conditions are not easily available in this case without recourse to some form of automated symbolic manipulation.

A great deal of insight into the present four-bar truss problem can, however, be obtained by considering a special symmetric case with

$$a_1 = a_2 \quad \text{and} \quad b_1 = b_2$$

which implies

$$\begin{aligned} &L_1 = L_2 = L_3 = L_4 = L \\ &\gamma_{12} = \gamma_{13} = \gamma_{14} = 1 \\ &c_{11} = c_{21} = c \quad \text{and} \quad s_{11} = s_{21} = s \end{aligned} \tag{22}$$

Use of Eqs. (22) into Eqs. (17-19) yields

$$\begin{aligned} A^* &= (A_1A_2 - A_3A_4)/(A_1+A_3)(A_2+A_4) \\ B^* &= (A_1A_4 - A_2A_3)/(A_1+A_3)(A_2+A_4) \end{aligned} \tag{23}$$

and

$$P_{cr} = \frac{E\alpha_1^3}{3\sqrt{3}}\left[(A_1+A_2+A_3+A_4) - \frac{(A_1A_2 - A_3A_4)}{(A_1+A_3)(A_2+A_4)}(A_1+A_2-A_3-A_4) - \frac{A_1A_4 - A_2A_3}{(A_1+A_3)(A_2+A_4)}(A_1-A_2-A_3+A_4)\right]$$

The last of Eqs. (23) may be simplified to

$$P_{cr} = \frac{4E\alpha_1^3}{3\sqrt{3}}\left[\frac{A_1A_2A_3 + A_1A_2A_4 + A_1A_3A_4 + A_2A_3A_4}{A_1A_2 + A_2A_3 + A_1A_4 + A_3A_4}\right] \tag{24}$$

Hence for the specialized truss, P_{cr} as given by Eq. (24) must be maximized subject to the constant-volume constraint

$$A_1L_1 + A_2L_2 + A_3L_3 + A_4L_4 = V_0 \tag{25}$$

The necessary optimality conditions

$$\begin{aligned} \frac{\partial P^*_{cr}}{\partial A_1} &= \frac{4E\alpha_1^3}{3\sqrt{3}}\frac{A_3^2}{(A_1+A_3)^2} - \lambda L = 0 \\ \frac{\partial P^*_{cr}}{\partial A_2} &= \frac{4E\alpha_1^3}{3\sqrt{3}}\frac{A_4^2}{(A_2+A_4)^2} - \lambda L = 0 \\ \frac{\partial P^*_{cr}}{\partial A_3} &= \frac{4E\alpha_1^3}{3\sqrt{3}}\frac{A_1^2}{(A_1+A_3)^2} - \lambda L = 0 \\ \frac{\partial P^*_{cr}}{\partial A_4} &= \frac{4E\alpha_1^3}{3\sqrt{3}}\frac{A_2^2}{(A_2+A_4)^2} - \lambda L = 0 \end{aligned} \tag{26}$$

have as their solution

$$A_1 = A_3 \quad \text{and} \quad A_2 = A_4 \tag{27}$$

One possible solution is of course

$$A_1 = A_2 = A_3 = A_4 = V_0/4L \tag{28}$$

Next, consider the strains ϵ_i, $i = 1,2,3,4$, in the four members at the optimum configuration. Use of Eqs. (27) into Eqs. (23) implies that

$$A^* = B^* = 0 \tag{29}$$

Expressions for the strains at the limit point provided by combining Eqs. (15), (18), and (20) are

$$\epsilon_1 = -(\alpha_1^2/3\sqrt{3})(1 - c_{11}A^* - s_{11}B^*) \tag{30a}$$

$$\epsilon_2 = -(\alpha_1^2/3\sqrt{3})(1 - c_{11}A^* + s_{21}B^*) \tag{30b}$$

$$\epsilon_3 = -(\alpha_1^2/3\sqrt{3})(1 + c_{21}A^* + s_{21}B^*) \tag{30c}$$

$$\epsilon_4 = -(\alpha_1^2/3\sqrt{3})(1 + c_{21}A^* - s_{11}B^*) \tag{30d}$$

which in light of Eqs. (29) reduce to simply

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = -\alpha_1^2/3\sqrt{3} \tag{30e}$$

This again implies constant-strain energy density in each of the four bars of the truss. In other words, every optimum solution will involve the same uniform stress distribution in all four bars and the existence of multiple optima for an indeterminate truss with uniform stress distribution can be re-established using Maxwell's lemma.[9] Maxwell's lemma can be stated as follows: "In any framework designed to transmit an assigned set of loads and to develop a stress f_t in every tie and a stress f_c in every strut, the difference $V_tf_t - V_cf_c$ is a constant, where V_t is the total volume of all the ties and V_c that of all the struts—ignoring, of course, the volumes of their end fittings." The proof of this lemma may be found in Ref. 9. The lemma holds for the case of large displacements so long as the strains are small.

The existence of multiple optima in the case of nonshallow trusses optimized with respect to a single displacement constraint on the basis of a linear analysis appears to be well known. Similar multiple optima can also be realized for the nonshallow version of the present four-bar truss if it is optimized with respect to a stability constraint on the basis of a linear eigenvalue analysis. As in the present case, the existence of multiple optima is simply the consequence of the fact that the optimality conditions reduce to a requirement for a uniform stress distribution in all the members of the truss. Maxwell's lemma then assures the existence of the infinity of optima.

Although, as indicated before, it is difficult to obtain explicit optimality conditions for the general four-bar truss in Fig. 2, maximization of P_{cr} subject to the volume constraint can be easily realized through recourse to the two options referred to earlier in the context of the two-bar truss. Numerical results again confirm that the criterion of constant-strain energy density holds true at optimum.

Of the two options, option 1 is very interesting since it allows the four distinct possibilities of choosing independent and dependent variables for initial guesses that satisfy the volume constraint. These are:

Case 1—A_1,A_2,A_3 independent; A_4 dependent.

Case 2—A_1,A_3,A_4 independent; A_2 dependent.

Case 3—A_1,A_2,A_4 independent; A_3 dependent.

Case 4—A_2,A_3,A_4 independent; A_1 dependent.

Nonuniqueness of the optimum designs becomes immediately obvious by using these four possibilities. The four independent solutions so obtained are summarized in Table 2. All of these solutions, although close, are distinct and not the result of round-off or limitations of computer precision. Furthermore, all of these solutions are nondegenerate; that is to say, none of the members are eliminated by virtue of their size tending to zero during the course of optimization.

Five optimum designs for the shallow four-bar truss are summarized in Table 2, the fifth solution having been obtained by the use of the VMCON algorithm. Many more distinct optimum designs can be obtained by beginning the optimization procedure with different initial guesses.

Table 3 shows how the measure of the strain energy density distribution in the case of a typical four-bar truss (case 3 of Table 2) gradually changes during the course of the optimization to attain uniformity at the optimum solution.

Finally, Tables 4 and 5 illustrate the progress of typical unconstrained optimizations of the two- and four-bar trusses, respectively, using the negative of the critical load P_{cr} as the objective function for minimization. It is clear from this table that as P_{cr} increases the potential energy π decreases to a minimum at the optimum. In fact, the same solutions for the two- and four-bar truss can also be obtained by using the potential energy as the objective function for minimization. Unconstrained optimization for the same four-bar truss of Table 5 using potential energy π as the objective function leads to an identical table of results with columns 5 and 7 of Table 5 interchanged. This confirms the duality between load maximization and potential energy minimization.

Conclusions

Both the two- and four-bar truss problems clearly indicate that, as long as the structures are shallow, the optimality criterion for maximum critical load at fixed volume reduces to that of a constant-strain energy density in the structure in spite of the nonlinearity. Furthermore, for such structures the possibility of more than one optimum design also exists. Finally, the problem of maximizing the critical load for a fixed volume is the dual of the problem of minimizing the weight for a given critical load.

References

[1]Stadler, W., "Stability Implications and the Equivalence of Stability and Optimality Conditions in the Optimal Design of Uniform Shallow Arches," *Proceedings of International Symposium on Optimum Structural Design*, 1981, pp. 3-3—3-10.

[2]Stadler, W., "Uniform Shallow Arches of Minimum Weight and Maximum Minimum Deflection," *Journal of Optimization Theory and Applications*, Vol. 23, No. 1, 1977, pp. 137-165.

[3]Christensen, E. N., "Optimal Design of Shallow Arches Against Buckling," Ph.D. Dissertation, Rensselear Polytechnic Institute, Troy, N.Y., 1975.

[4]Caldwell, H. M., "Optimization of Shallow Arches Against Snap-Through Buckling," Ph.D. Dissertation, Georgia Institute of Technology, Atlanta, 1977.

[5]Thompson, J. M. T. and Hunt, G. W., *A General Theory of Elastic Stability*, John Wiley & Sons, New York, 1973.

[6]Bergan, P. and Soreida, T., "A Comparative Study of Different Numerical Solution Techniques as Applied to a Nonlinear Structural Problem," *Computer Methods in Applied Mechanics and Engineering*, Vol. 2, 1973, pp. 185-201.

[7]Shanno, D. F. and Phua, K. H., "Algorithm 500, Minimization of Unconstrained Multivariate Functions," ACM TOMS, Vol. 2, 1976, pp. 87-94.

[8]Crane, R. L., Hillstrom, K. E., and Minkoff, M., "Solution of the General Nonlinear Programming Problem with Subroutine VMCON," Argonne National Laboratory, Argonne, Ill., Rept. ANL-80-64, 1980.

[9]Cox, H. L., *The Design of Structures of Least Weight*, Pergamon Press, New York, 1965.